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EDGE CRACK AT THE BOUNDARY OF DIFFERENT MEDIA

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There is considered the plane problem of elasticity theory concerning the equilibrium of an elastic half-plane consisting of two materials with a rectilinear edge crack located at the interface of these materials and emerging into the load free boundary of the half-plane. The problem mentioned reduces to a Riemann boundary value problem for two pairs of functions. Under the condition that the sum of the tripled compression modulus and the shear modulus is identical for both materials, a solution is given by an exact analytical method and the stress intensity factors at the vertex of the crack are calculated.

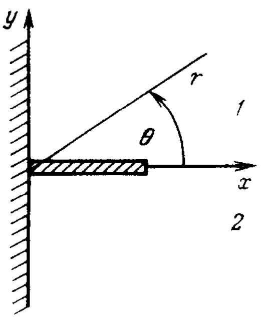

Fig. 1

Let us consider an elastic half-plane $x > 0$ composed of two materials: for $y > 0$ with the subscript 1 and for $y < 0$ with the subscript 2. At $y = 0$, $x < 1$ on the boundary between the media there is a crack at whose edges a given normal load $\sigma_y = -\sigma$, $\tau_{xy} = 0$ is applied (see Fig. 1). The half-plane boundary $x = 0$ is load free. The stresses vanish at infinity.

Let us write the equilibrium equations, the strain compatibility condition, and the boundary conditions in the polar coordinates $r\theta$

$$r\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{r\theta}}{\partial\theta} + \sigma_r - \sigma_\theta = 0 \tag{1}$$

$$\frac{\partial\sigma_\theta}{\partial\theta} + r\frac{\partial\tau_{r\theta}}{\partial r} + 2\tau_{r\theta} = 0, \quad \Delta(\sigma_r + \sigma_\theta) = 0$$

$$\theta = 0, \quad [\sigma_\theta] = [\tau_{r\theta}] = 0 \tag{2}$$

$$\theta = \pm\pi/2, \quad \sigma_\theta = \tau_{r\theta} = 0$$

$$\theta = 0, \quad 0 < r < 1, \quad \sigma_\theta = -\sigma, \quad \tau_{r\theta} = 0 \tag{3}$$

$$\theta = 0, \quad r > 1, \quad [u_\theta] = [u_r] = 0$$

$$r \to \infty, \quad \sigma_\theta \to 0, \quad \tau_{r\theta} \to 0, \quad \sigma_r \to 0 \tag{4}$$

(σ_r, σ_θ, $\tau_{r\theta}$ are stresses and u_θ, u_r are displacements).

From physical considerations, the stresses will be bounded as $r \to 0$ and will behave as $1/r^2$ as $r \to \infty$.

Applying the Mellin transform with the complex parameter p [1] to (1) and satisfying the boundary conditions (2), we arrive at the following expression for the transform $\bar{\sigma}_\theta(p, \theta)$:

$$\bar{\sigma}_\theta(p, \theta) = A_1 \sin(p+1)\theta + A_2 \sin(p-1)\theta + A_3 \cos(p+1)\theta + A_4 \cos(p-1)\theta \quad (5)$$

$$A_i = \begin{cases} A_i^+, & 0<\theta<\pi/2 \\ A_i^-, & -\pi/2<\theta<0 \end{cases} \quad (i=1, 2, 3, 4)$$

$$A_2^\pm(p) = \left[-p\cos^2\frac{p\pi}{2} A_1^\pm(p) + \left(p^2 - \sin^2\frac{p\pi}{2}\right) A_1^\mp(p)\right] \times \left[(p-1)\left(p - \sin^2\frac{p\pi}{2}\right)\right]^{-1}$$

$$A_3^\pm(p) = \pm\left[-\left(p^2 + \cos p\pi \sin^2\frac{p\pi}{2}\right) A_1^\pm(p) + A_1^\mp(p) \times \left(p^2 - \sin^2\frac{p\pi}{2}\right)\right]\left[\sin p\pi\left(p - \sin^2\frac{p\pi}{2}\right)\right]^{-1}$$

$$A_4^\pm(p) = \mp\left\{\left[(p-1)(p^2-1) + (p+\cos p\pi)\left(2p\cos^2\frac{p\pi}{2} + p^2 - 1\right)\right] A_1^\pm(p) - 2\left(p^2 - \sin^2\frac{p\pi}{2}\right)(p+\cos p\pi) A_1^\mp(p)\right\} \times \left[2(p-1)\sin p\pi\left(p - \sin^2\frac{p\pi}{2}\right)\right]^{-1}$$

($A_1^+(p)$ and $A_1^-(p)$ are unknown functions of p).

Let us introduce the functions (taking the boundary conditions (3) into account)

$$\Phi^-(p) = \frac{E_1}{4(1-\nu_1^2)} \int_0^1 \left[\frac{\partial u_\theta}{\partial r}\right]\Big|_{\theta=0} r^p\, dr = \frac{E_1}{4(1-\nu_1^2)} \left[\overline{\frac{\partial u_\theta}{\partial r}}\right]\Big|_{\theta=0} \quad (6)$$

$$\Psi^-(p) = \frac{E_1}{4(1-\nu_1^2)} \int_0^1 \left[\frac{\partial u_r}{\partial r}\right]\Big|_{\theta=0} r^p\, dr = \frac{E_1}{4(1-\nu_1^2)} \left[\overline{\frac{\partial u_r}{\partial r}}\right]\Big|_{\theta=0}$$

$$U^+(p) = \int_1^\infty \sigma_\theta(r, 0)\, r^p\, dr = \bar{\sigma}_\theta(p, 0) + \frac{\sigma}{p+1}$$

$$V^+(p) = \int_1^\infty \tau_{r\theta}(r, 0)\, r^p\, dr = \bar{\tau}_{r\theta}(p, 0)$$

(E_1, E_2 and ν_1, ν_2 are the Young's moduli and Poisson's ratios).

Eliminating the functions $A_1^+(p)$ and $A_1^-(p)$ in (6) by using Hooke's law, we arrive at a Wiener — Hopf equation:

$$\varphi^-(p) = A \operatorname{ctg}\frac{p\pi}{2} G(p)\left[\varphi^+(p) + C(p)\right] \quad (7)$$

$$G(p) = \begin{Vmatrix} g_0 & g_- \\ g_+ & g_0 \end{Vmatrix}, \quad g_0 = A^{-1}\frac{k_2+1}{2} b(p)$$

$$g_\pm = \pm A^{-1} \operatorname{tg}\frac{p\pi}{2}\left[k_1 + \frac{k_2-1}{2}\left(1 \pm p\sin^{-2}\frac{p\pi}{2}\right) b(p)\right]$$

$$k_1 = \frac{k-1}{4(1-\nu_1)}, \quad k_2 = \frac{1-\nu_2}{1-\nu_1}k, \quad k = \frac{E_1(1+\nu_2)}{E_2(1+\nu_1)}$$
$$A = \sqrt{(k_1+1)(k_2-k_1)}, \quad b(p) = \sin^2\frac{p\pi}{2}\left(p^2 - \sin^2\frac{p\pi}{2}\right)^{-1}$$
$$C(p) = \left(-\frac{\sigma}{p+1}, 0\right), \quad \varphi^-(p) = (\Phi^-(p), \Psi^-(p))$$
$$\varphi^+(p) = (U^+(p), V^+(p))$$

We assume that the elastic constants are connected by the relationship $2k_1 + 1 = k_2$. Physically this corresponds to the assumption that the sum of thrice the compression modulus plus the shear modulus is identical for both materials.

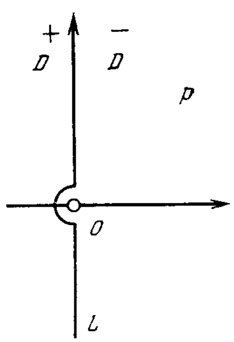

Fig. 2

Let us consider a contour consisting of the imaginary axis, with the exception of a small symmetric segment around the origin, and a left semi-circle of small radius with center at the origin (Fig. 2) in the plane of the complex variable p. The domain to the left and right of the contour will be denoted by D^+ and D^-, respectively. The matrix $G(p)$ in (7) has the following properties

$$G(p) = b(p)\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix} + 2\chi\frac{pb(p)}{\sin p\pi}\begin{Vmatrix}0 & p-1\\ -p-1 & 0\end{Vmatrix}$$
$$\chi = (1-k_2)/(1+k_2)$$

Let $p = it$ $(-\infty < t < \infty)$ and $\Delta(p)$ be the determinant of the matrix $G(p)$, then

$$0 < \Delta(it) = s^2(t)\left[1 - 4\chi^2\frac{t^2(t^2+1)}{\operatorname{sh}^2 t\pi}\right]$$
$$\left(s(t) = \operatorname{sh}^2\frac{t\pi}{2} \Big/ \left(\operatorname{sh}^2\frac{t\pi}{2} - t^2\right)\right)$$

is an even function, $\lim \Delta(it) = 1$ as $t \to \infty$, and $\Delta(p)$ is an analytic function positive at the point $p = 0$. Therefore, $\Delta(p) \neq 0$ for $p \in L$.

Let λ_1 and λ_2 be eigennumbers of the matrix G.

Since $\lim \Delta(it) = 1$ for $t \to \pm\infty$, then

$$\varkappa_\Delta = \frac{1}{4\pi i}[\ln(\lambda_1\lambda_2)]\big|_L = 0$$

Since

$$\lambda_{1,2}(it) = -s(t) \pm \left[4\chi^2\frac{t^2(t^2+1)s^2(t)}{\operatorname{sh}^2 t\pi}\right]^{1/2}, \quad \varepsilon = \frac{1}{2}\ln\frac{\lambda_1}{\lambda_2}$$

then $0 > \varepsilon(it)$ is an even function, where $\lim \varepsilon(it) = 0$ as $t \to \pm\infty$.

Therefore

$$\varkappa_\varepsilon = \frac{1}{4\pi i}\left[\ln \frac{\lambda_1}{\lambda_2}\right]\Big|_L = 0$$

According to the theorem presented in [2], we obtain from the properties of the matrix $G(p)$

$$G(p) = \frac{X^+(p)}{X^-(p)}, \quad X(p) = F(p)\begin{Vmatrix} x_0 & x_+ \\ x_- & x_0 \end{Vmatrix} \quad (p \in L)$$

$$x_0 = \operatorname{ch}\left[\sqrt{1-p^2}\beta(p)\right], \quad x_\pm = \frac{+p-1}{\sqrt{1-p^2}}\operatorname{sh}\left[\sqrt{1-p^2}\beta(p)\right]$$

$$F(p) = \exp\left[\frac{1}{4\pi i}\int_L \frac{\ln\Delta(t)}{t-p}dt\right], \quad \beta(p) = \frac{1}{2\pi i}\int_L \frac{\varepsilon(t)}{\sqrt{f(t)}}\frac{dt}{t-p}$$

We write (7) as follows:

$$\frac{1}{2AK^-(p/2)}X^-(p)\varphi^-(p) + M^-(p) = \frac{K^+(p/2)}{p}X^+(p)\varphi^+(p) + M^+(p)$$

$$(p \in L)$$

$$K^\pm(p/2) = \Gamma(1 \mp p/2)/\Gamma(^1/_2 \mp p/2)$$

$$\frac{1}{2\pi i}\int_L \frac{K^+(t/2)}{t}X^+(t)C(t)\frac{dt}{t-p} = \begin{cases} M^+(p), & p \in D^+ \\ M^-(p), & p \in D^- \end{cases}$$

Using the relationships near the tip of the crack [3] and a theorem of Abelian type [4], we obtain

$$U^+(p) \sim \frac{K_\mathrm{I}}{\sqrt{2}}\frac{1}{\sqrt{-p}}, \quad V^+(p) \sim \frac{K_\mathrm{II}}{\sqrt{2}}\frac{1}{\sqrt{-p}} \quad (p \to \infty) \tag{8}$$

$$\left[\sigma_\theta(r,0) \sim \frac{K_\mathrm{I}}{\sqrt{2\pi(r-1)}}, \quad \tau_{r\theta}(r,0) \sim \frac{K_\mathrm{II}}{\sqrt{2\pi(r-1)}} \quad (r \to 1+0)\right]$$

Here K_I, K_II are stress intensity factors at the crack vertex.
On the basis of (8), the solution of the Wiener — Hopf equation has the form

$$\varphi^+(p) = -[p\,/\,K^+(p\,/\,2)][X^+(p)]^{-1}M^+(p)$$
$$\varphi^-(p) = -2AK^-(p\,/\,2)[X^-(p)]^{-1}M^-(p) \tag{9}$$

Let us find the stress intensity factors at the crack vertex. By using residue theory, we obtain from (9)

$$U^+(p) \sim [\sigma\sqrt{\pi/2}F^+(-1)\cos q]\,/\sqrt{-p} \qquad (p \to \infty) \tag{10}$$
$$V^+(p) \sim [-\sigma\sqrt{\pi/2}F^+(-1)\sin q]\,/\sqrt{-p}$$

$$q = \frac{1}{2\pi i}\int_L \frac{\varepsilon(p)}{\sqrt{f(p)}}dp = \frac{1}{\pi}\int_0^\infty \frac{\varepsilon(it)}{\sqrt{t^2+1}}dt$$

$$F^+(-1) = \exp\left[\frac{1}{4\pi i}\int_L \frac{\ln\Delta(p)}{p+1}\,dp\right] = \exp\left[\frac{1}{2\pi}\int_0^\infty \frac{\ln\Delta(it)}{t^2+1}\,dt\right]$$

Comparing the asymptotics in (10) and (8), we find

$$K_I = \sigma\sqrt{\pi}F^+(-1)\cos q, \quad K_{II} = -\sigma\sqrt{\pi}F^+(-1)\sin q \tag{11}$$

Presented below are the dependences $\mu_1 = K_I / \sigma\sqrt{\pi}$ and $\mu_2 = K_{II} / \sigma\sqrt{\pi}$ on k for $\nu_1 = {}^1/_3$

k	0.34	0.5	1	2	4	8
μ_1	1.1171	1.1193	1.1215	1.1185	1.1102	1.0994
$100\,\mu_2$	0.6861	0.3383	0	0.4722	1.7693	3.2936

If $k_2 = 1$, $k_1 = 0$ (homogeneous medium), the result ($k = 1$) agrees with one known [5].

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